Conformal Transformations: The Covariant Approach Perspective

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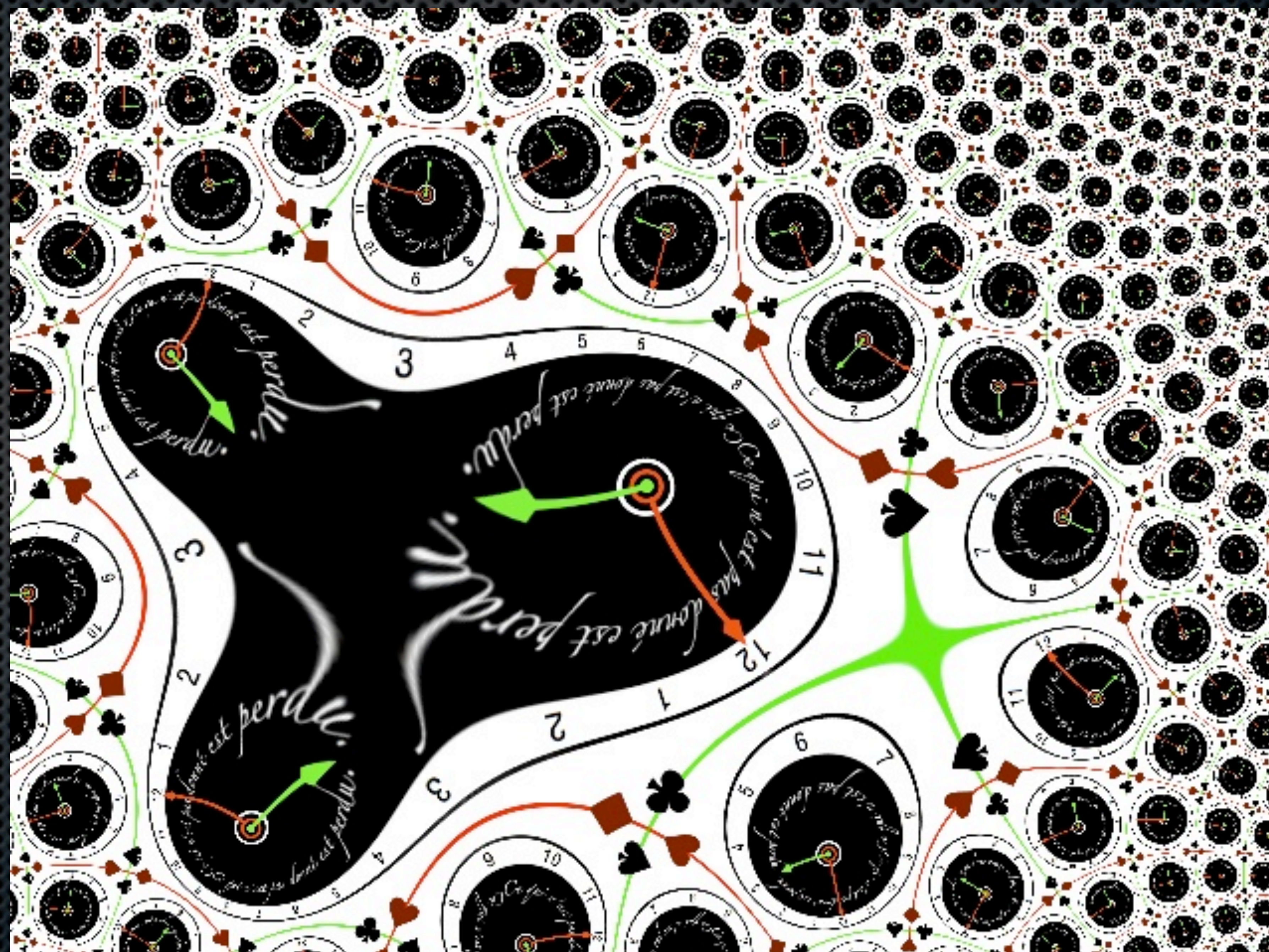


"The Invisible Universe"

Paris 03/07/2009

Outline

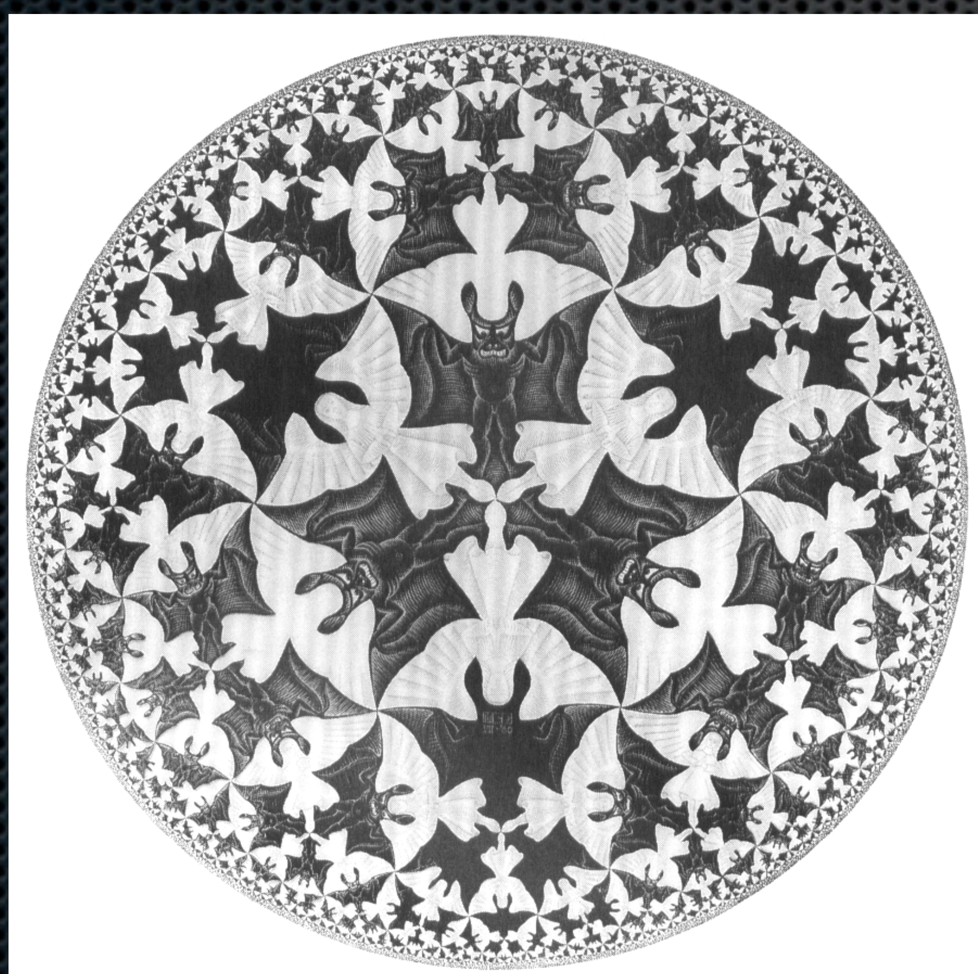
- Conformal transformations
- Conformal transformations and alternative gravity (specifically f(R)-gravity)
- 1+3 conformal transformation
- transformation of the 1+3 covariant approach
- Transformation of the CoGI theory of perturbations for f(R)-gravity
- An example with a toy model

Conformal transformations (CT)

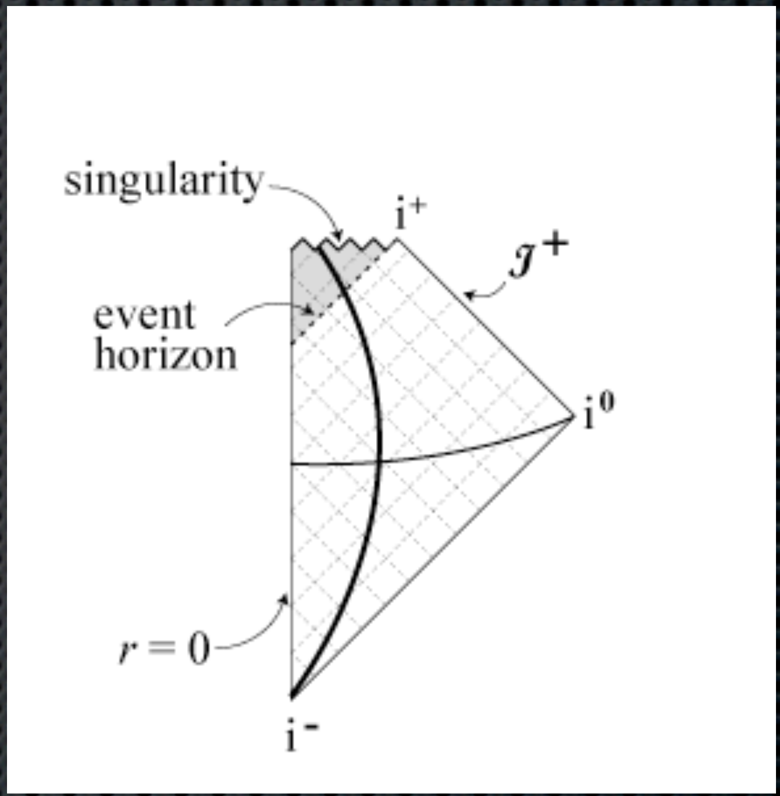
Conformal transformations in Riemannian geometry are defined as

$$g_{ab} \to \bar{g}_{ab} = \Upsilon(t,x) g_{ab}$$

with $\Upsilon > 0$. They preserve angles and the signature of the manifold.

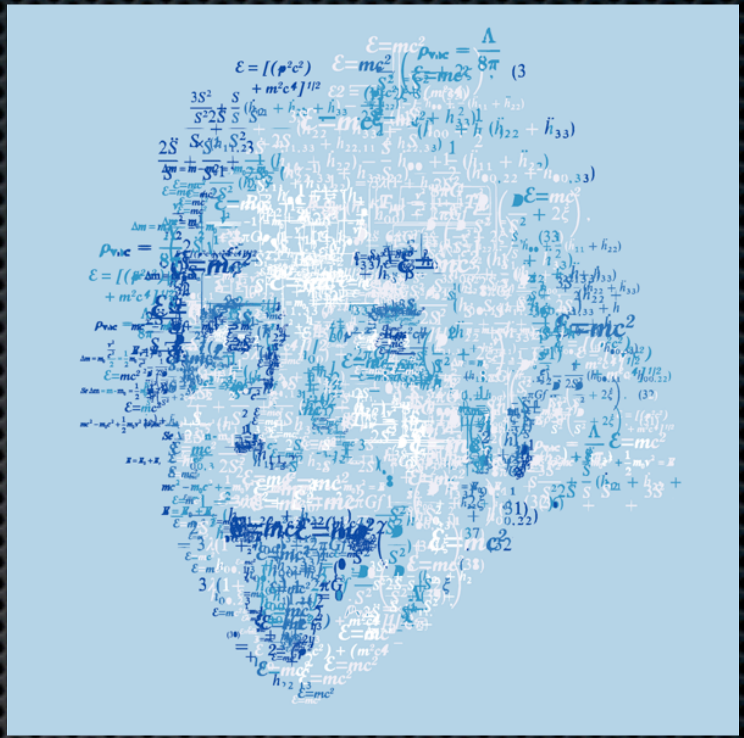


- CTs are used in many different fields including Relativity



- They also have interesting applications in alternative gravity...

Fourth Order Gravity



In homogeneous and isotropic spacetimes a general Lagrangian for fourth order gravity in presence of matter is

$$L = \sqrt{-g}\left[f(R) + 2\mathcal{L}_m(\beth, g^{ab})\right]$$

varying with respect to the metric gives

$$F(R)R_{ab} - \frac{1}{2}g_{ab}f(R) = F(R)_{;cd}\left(g_a^c g_b^d - g_{ab}g^{cd}\right) + T_{ab}^m$$

where

$$T_{ab}^m = \frac{2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\mathcal{L}_m)}{\delta g_{ab}}$$

and F denotes the derivative of f with respect to $R.$

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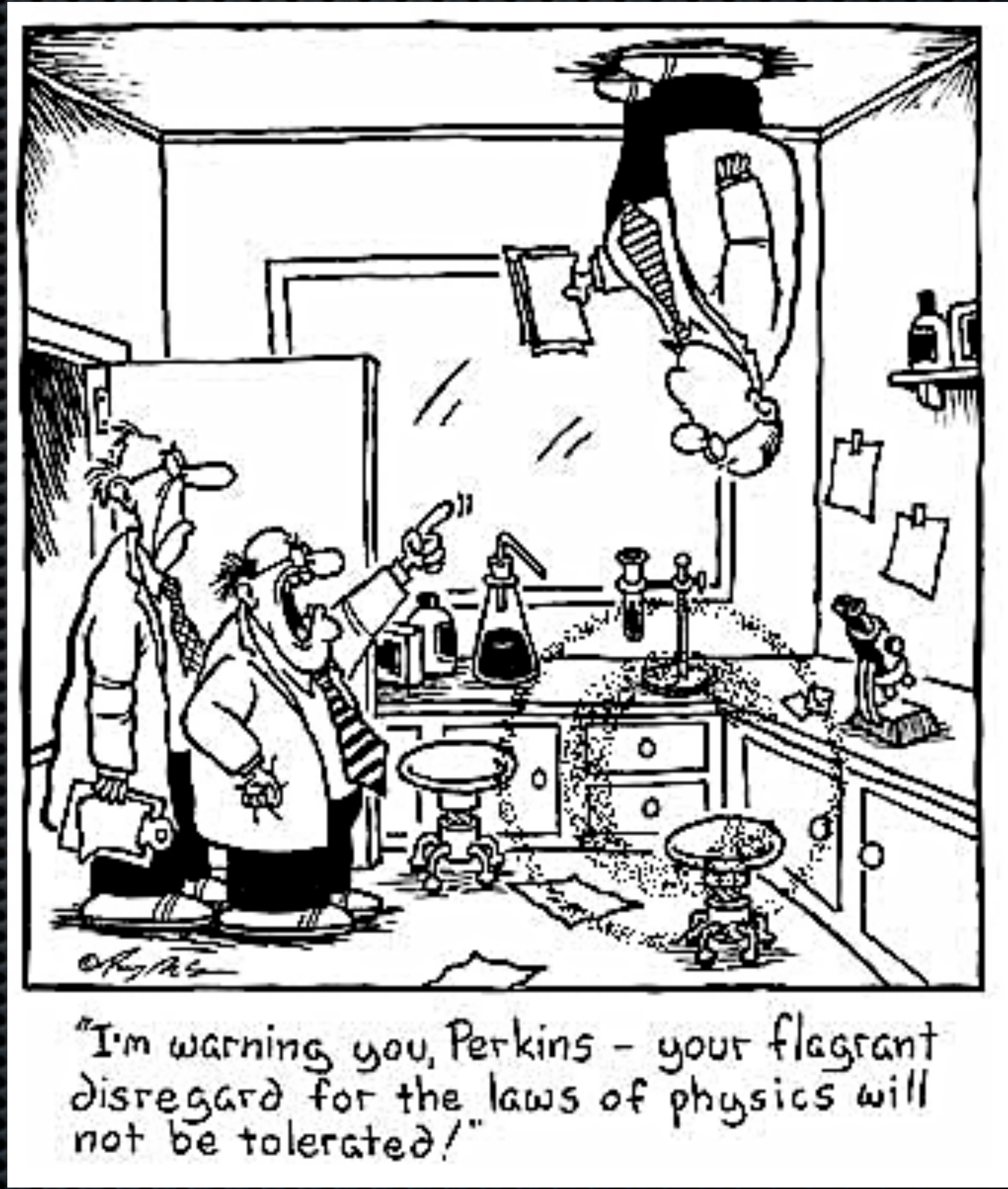
f(R)-gravity and CT

- These theories emerge as low energy limit of fundamental theories, like M-theory or supergravity
- They have recently been used to model Dark Energy with some success...
- However one of the issues of these theories is their complexity due to the non linearities and the additional degrees of freedom.



- CTs are important for these models: they reduce complex theories of gravity to Einstein gravity plus a minimally coupled scalar field, which is easier to treat.

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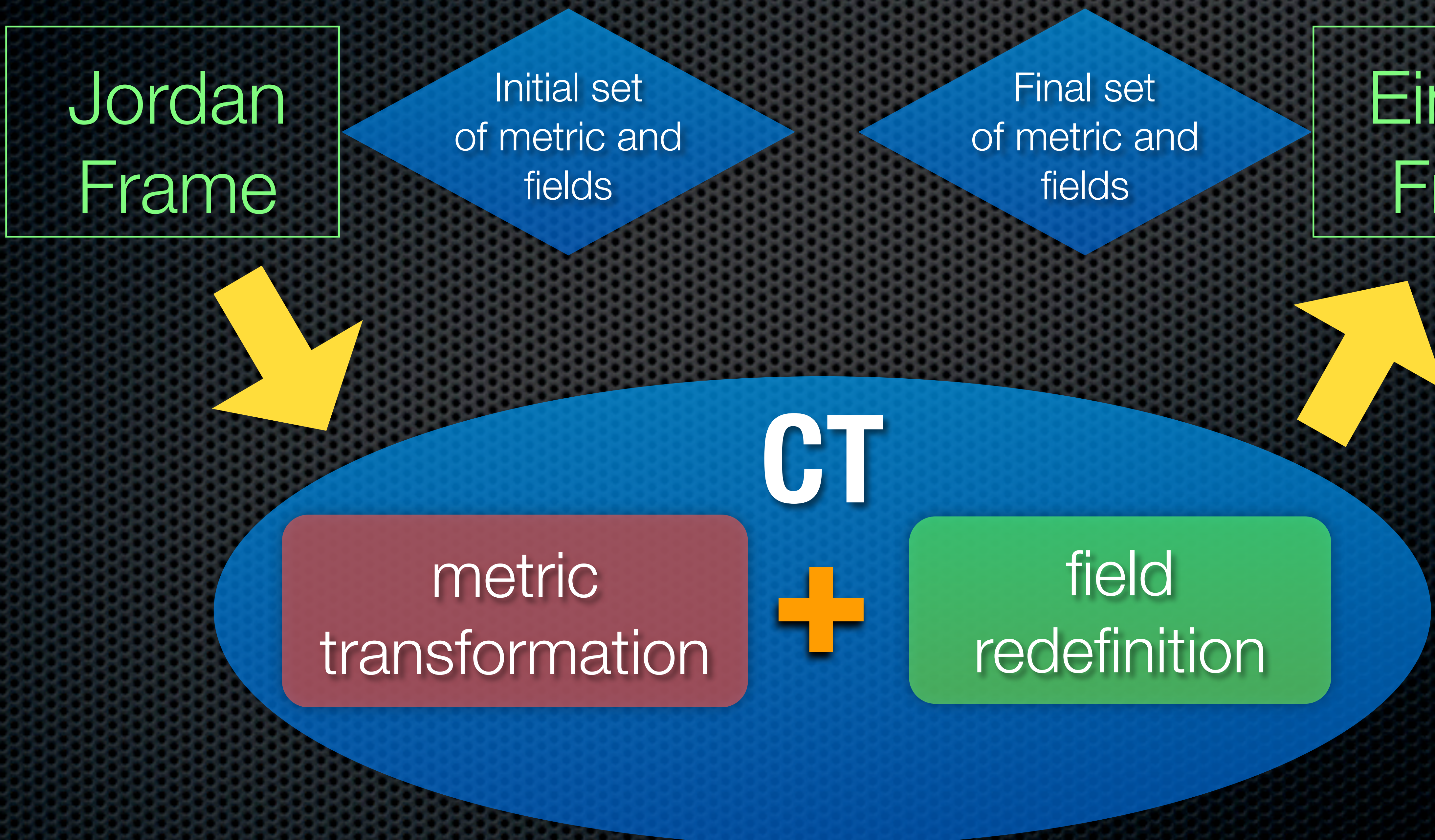
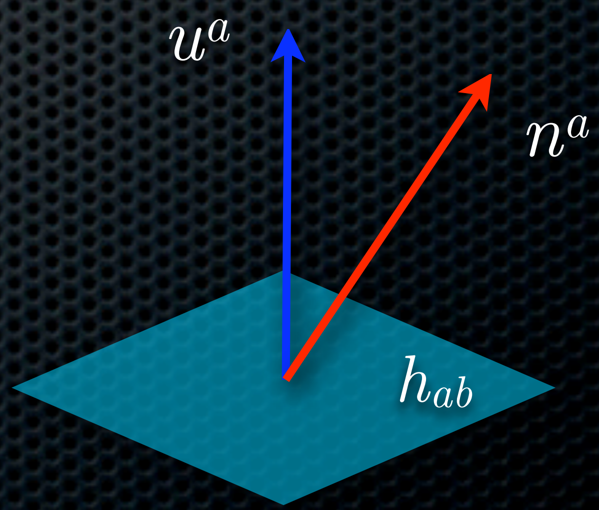
CT in alternative gravity

- In alternative gravity however a conformal transformation is not just the geometrical transformation showed above...



The 1+3 covariant approach
(in brief)

- A time-like flow u^a is introduced, e.g., the fluid flow of galaxy cluster or any fluid flow in cosmology or astrophysics.
- The remaining spatial freedom is left untouched.
- From u^a we construct the projection onto surfaces orthogonal to the flow: $h_{ab} = g_{ab} + u_a u_b$
- Three-volume form: $\eta_{abc} = u^d \eta_{abcd}$
- Covariant convective derivative and projected derivative:



$$\dot{X}_c = u^e \nabla_e X_c \,, \qquad \widetilde{\nabla}_e X_c = h^p{}_c \, h^r{}_e \nabla_r X_q$$

- Kinematics of u^a gives the kinematics of the model

$$\nabla_a u_b = -u_a \overbrace{a_b}^{\text{acceleration}} + \frac{1}{3}\overbrace{\Theta}^{\text{expansion}} h_{ab} + \overbrace{\sigma_{ab}}^{\text{shear}} + \overbrace{\omega_{ab}}^{\text{vorticity}}$$

- A general energy-momentum tensor can be decomposed relative to u^a and h_{ab}

$$T_{ab} = \overbrace{\mu}^{\text{energy density}} u_a u_b + \overbrace{q_a}^{\text{flux}} u_b + u_a \overbrace{q_b}^{\text{flux}} + \overbrace{p}^{\text{pressure}} h_{ab} + \overbrace{\pi_{ab}}^{\text{anisotropic pressure}}$$

- Obtain a set of propagation and constraint equations for these variables based on Ricci and Bianchi identities: the 1+3 equations.

- Covariant variables have an clear physical interpretation and can be used to investigate both the exact models and perturbations.

1+3 Conformal Transformation

- Let us now see the conformal transformation in this formalism. We start with the geometric transformation.

$$g_{ab} \to \bar{g}_{ab} = \Upsilon(t,x) g_{ab} \quad \Rightarrow \quad \begin{cases} h_{ab} \to \bar{h}_{ab} = \boxed{\Upsilon h_{ab}} \\ \boxed{u_a \to \bar{u}_a} = \boxed{\sqrt{\Upsilon} u_a} \end{cases}$$

change of observer

rod length variation

clock recalibration

acceleration

- With this we can transform the derivatives...

$$X^{\dagger}_a = \frac{1}{\sqrt{\Upsilon}} \left[\dot{X}_a - \frac{1}{\Upsilon} u^b X_{(b} \nabla_{a)} \Upsilon + \frac{1}{2\Upsilon} u_a X^r \nabla_r \Upsilon \right] ,$$

$$\widetilde{\widetilde{\nabla}}_e X_a = \widetilde{\nabla}_e X_a - \frac{1}{\Upsilon} X_{(e} \widetilde{\nabla}_{a)} \Upsilon + \frac{1}{2\Upsilon} h_{ea} X^r \nabla_r \Upsilon .$$

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Thus the transformation of the kinematic quantities is

$$\bar{\Theta} = \frac{1}{\sqrt{\Upsilon}}\left(\Theta + \boxed{\frac{3}{2}\frac{\dot{\Upsilon}}{\Upsilon}}\right)\,, \qquad \bar{a}_b = a_b + \frac{1}{2}\boxed{\frac{\overline{\nabla}_b\Upsilon}{\Upsilon}}\,,$$

$$\bar{\sigma}_{ab} = \boxed{\sqrt{\Upsilon}\sigma_{ab}}\,, \qquad \bar{\omega}_{ab} = \boxed{\sqrt{\Upsilon}\omega_{ab}}$$

and the one of the thermodynamical ones

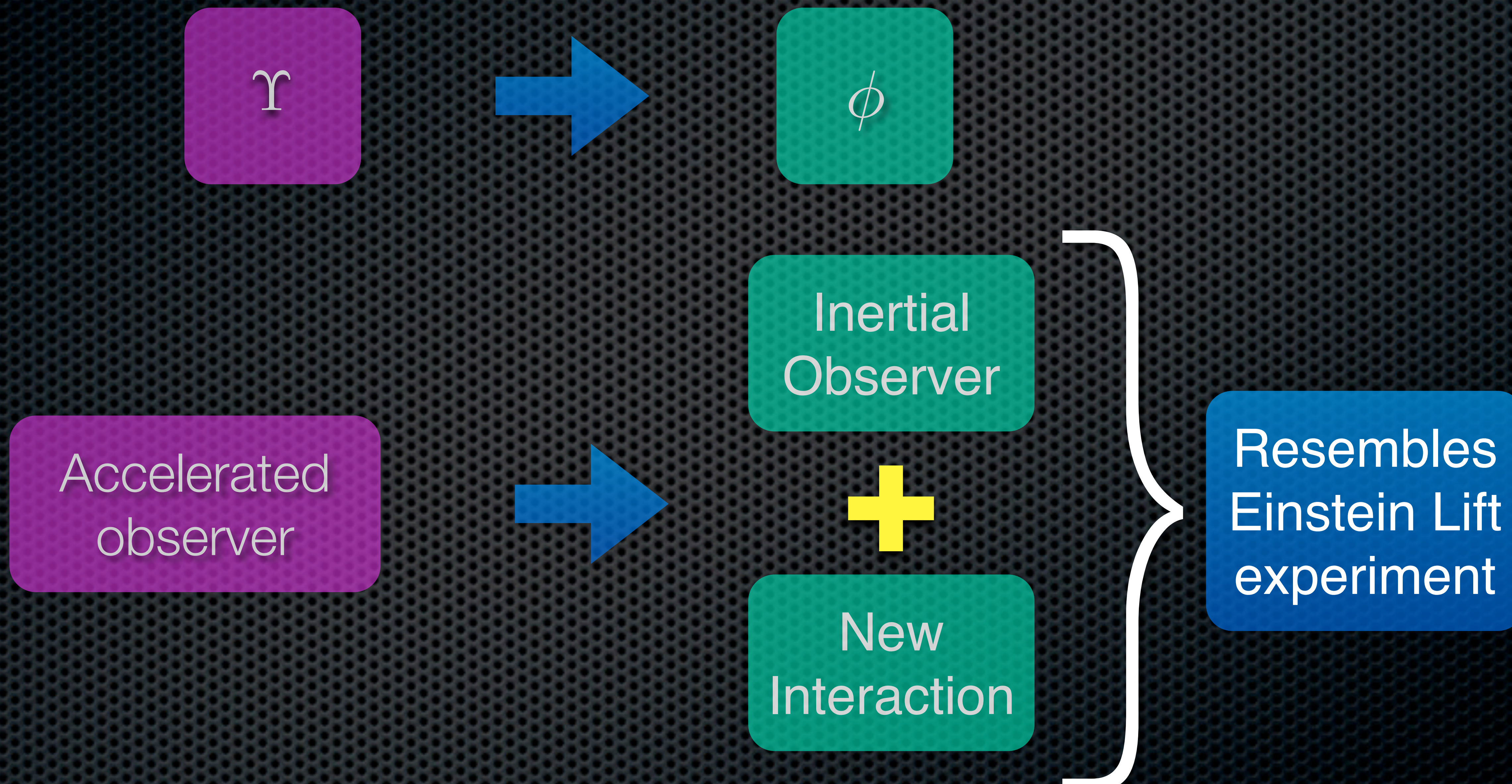
$$\bar{\mu} = T_{ab}\bar{u}^a\bar{u}^b = \boxed{\frac{\mu}{\Upsilon}}\,, \qquad \bar{p} = \frac{1}{3}T_{ab}\bar{h}_{ab} = \boxed{\frac{p}{\Upsilon}}\,,$$

$$\bar{q}_a = -T_{bc}\,\bar{u}^b\,\bar{h}^{ca} = \boxed{\frac{q_a}{\sqrt{\Upsilon}}}\,, \qquad \bar{\pi}_{ab} = T_{cd}\,\bar{h}^c{}_{\langle a}\,\bar{h}^d{}_{b\rangle} = \boxed{\pi_{ab}}$$

- With this we can transform all of the 1+3 equations...

1+3 Conformal Transformation

- Let us look now at the field redefinition.



- The transformation of the kinetic quantity Υ in the matter field ϕ is the core of the physical change in the CTs.

1+3 CT and f(R)-gravity

- When one performs the CT in f(R)-gravity...

$$F \rightarrow \Upsilon \rightarrow \phi$$

- The accelerated observer dynamics is such to compensate most the effects of fourth order gravity
- Only terms that can be associated to the contribution of a scalar field are left...
- ...and of course the non minimal coupling with matter

Cosmological Perturbations

The 1+3 covariant approach allows to construct a covariant and gauge invariant theory of perturbations.

Exact 1+3 equations valid in any spacetime.

→ Choose background spacetime.

↓ Variables that vanish in chosen background are O(1) and GI.

← Linearize by dropping all terms that are O(2) and higher.

Scalar Perturbations variables

The basic set of scalar inhomogeneity variables is:

$$\Delta^m = \frac{S^2}{\mu^m}\widetilde{\nabla}^2\mu^m\,, \qquad Z = S^2\widetilde{\nabla}^2\Theta\,, \qquad C = S^4\widetilde{\nabla}^2\tilde{R},$$

but one might need others, depending on the degrees of freedom of the theory analyzed. Also they are not independent.

Let see how they behave under CT

$$\overline{\Delta} = \Upsilon\left(\Delta - \boxed{S^2\frac{\widetilde{\nabla}^2\Upsilon}{\Upsilon}}\right),$$

$$\overline{Z} = \sqrt{\Upsilon}\left(Z - \frac{1}{2}S^2\Theta\frac{\widetilde{\nabla}^2\Upsilon}{\Upsilon} - \frac{9}{4}S^2\frac{\dot{\Upsilon}}{\Upsilon}\frac{\widetilde{\nabla}^2\Upsilon}{\Upsilon} + \frac{3}{2}S^2\frac{\widetilde{\nabla}^2\dot{\Upsilon}}{\Upsilon}\right),$$

$$\overline{C} = \sqrt{\Upsilon}\left[C - \boxed{2S^2 Z\frac{\dot{\Upsilon}}{\Upsilon} + S^3\left(8\Theta\frac{\dot{\Upsilon}}{\Upsilon} + 9\frac{\dot{\Upsilon}^2}{\Upsilon^2} - 2R_3\right)\frac{\widetilde{\nabla}^2\Upsilon}{\Upsilon} - S^3\left(2\Theta + \frac{3\dot{\Upsilon}}{\Upsilon}\right)\frac{\widetilde{\nabla}^2\dot{\Upsilon}}{\Upsilon}}\right]$$

Scalar Perturbations of f(R)-gravity
(Jordan Frame)

Let us now compare the perturbations in the two frames. The H-I background in JF can be described by

$$\Theta^2 = 3\frac{\mu^m}{F} + 3\mu^R - \frac{3R_3}{2}\,,$$

$$\dot{\Theta} + \tfrac{1}{3}\Theta^2 + \tfrac{1}{2F}(\mu^m + 3p^m) + \tfrac{1}{2}(\mu^R + 3p^R) = 0\,,$$

$$\dot{\mu}^m + \Theta\,(\mu^m + p^m) = 0\,,$$

$$\dot{\mu}^R + \Theta\,(\mu^R + p^R) - \mu^m\frac{F'}{F^2}\dot{R} = 0\,,$$

Where

$$\mu^R = \frac{1}{F}\left[\frac{1}{2}(RF - f) - \Theta\dot{F} + \tilde{\nabla}^2 F\right]\,,$$

$$p^R = \frac{1}{F}\left[\frac{1}{2}(f - RF) + \ddot{F} + \frac{2}{3}\Theta\dot{F} - \frac{2}{3}\tilde{\nabla}^2 F - a_b\tilde{\nabla}^b F\right]\,,$$

Scalar Perturbations of f(R)-gravity (JF)

The perturbations variables we need are

$\Delta,\ Z,\ C$ **+** $\mathcal{R} = S^2\widetilde{\nabla}^2 R\,,\quad \Re = S^2\widetilde{\nabla}^2 \dot{R}$

The perturbation equations can be then written as

$$\ddot{\Delta}_m^{(\ell)} + \mathcal{A}\dot{\Delta}_m^{(\ell)} + \mathcal{B}\Delta_m^{(\ell)} = \mathcal{C}\mathcal{R}^{(\ell)} + \mathcal{D}\dot{\mathcal{R}}^{(\ell)}\,,$$
$$F'\ddot{\mathcal{R}}^{(\ell)} + \mathcal{E}\dot{\mathcal{R}}^{(\ell)} + \mathcal{F}\mathcal{R}^{(\ell)} = \mathcal{G}\Delta_m^{(\ell)} + \mathcal{H}\dot{\Delta}_m^{(\ell)}$$

- The coefficients are very complicated functions of first order quantities.
- This system present many difference with respect to the GR counterpart.

Scalar Perturbations of f(R)-gravity
(Einstein Frame)

The H-I background can be described by

$$\bar{\Theta}^2 = 3\bar{\mu}^m e^{\left(-\sqrt{2/3}\phi\right)} + 3\mu^\phi - \frac{3\bar{R}_3}{2} ,$$

$$\bar{\Theta}^\dagger + \tfrac{1}{3}\bar{\Theta}^2 + \tfrac{1}{2}(\bar{\mu}^m + 3\bar{p}^m)e^{\left(-\sqrt{2/3}\phi\right)} + \tfrac{1}{2}(\bar{\mu}^\phi + 3\bar{p}^\phi),$$

$$\bar{\mu}_m^\dagger + \bar{\Theta}\left(\bar{\mu}^m + \bar{p}^m\right) - \sqrt{\frac{2}{3}}\,\bar{\mu}^m\phi^\dagger - \frac{1}{\sqrt{6}}(3\bar{p}^m - \bar{\mu}^m)\phi^\dagger = 0 ,$$

$$\bar{\Box}\phi - W'(\phi) = \frac{1}{\sqrt{6}}(3\bar{p}^m - \bar{\mu}^m)\, e^{\left(-\sqrt{2/3}\phi\right)}$$

Where

$$\mu^\phi = \frac{1}{2}(\phi^\dagger)^2 + \frac{1}{2}\tilde{\bar{\bar{\nabla}}}{}^a\phi\tilde{\bar{\bar{\nabla}}}_a\phi + W(\phi) ,$$

$$p^\phi = \frac{1}{2}(\phi^\dagger)^2 - \frac{1}{6}\tilde{\bar{\bar{\nabla}}}{}^a\phi\tilde{\bar{\bar{\nabla}}}_a\phi - W(\phi) ,$$

Scalar Perturbations of f(R)-gravity (EF)

The perturbations variables we need are

$\bar{\Delta}, \bar{Z}, \bar{C}$ + $$\bar{\Phi} = S^2 \widetilde{\bar{\nabla}}^2 \phi\,, \qquad \bar{\Psi} = S^2 \widetilde{\bar{\nabla}}^2 \phi^\dagger$$

The perturbation equations can be then written as

$$\bar{\Delta}^{\dagger\dagger}_{(\ell)} = \mathcal{A}\,\bar{\Delta}^{\dagger}_{(\ell)} + \mathcal{B}\,\Delta_{(\ell)} + \mathcal{C}\,\bar{\Phi}_{(\ell)} + \mathcal{D}\,\bar{\Phi}^{\dagger}_{(\ell)}\,,$$
$$\bar{\Phi}^{\dagger\dagger}_{(\ell)} = \mathcal{E}\,\bar{\Phi}^{\dagger}_{(\ell)} + \mathcal{F}\,\bar{\Phi}_{(\ell)} + \mathcal{G}\,\bar{\Delta}_{(\ell)} + \mathcal{H}\,\bar{\Delta}^{\dagger}_{(\ell)}$$

- The equations have a similar differential structure and the same scale dependence
- They differ in the properties of the long wavelength limit as it was anticipated by the transformation of the variable C.

A worked example: R^n-gravity.

Let us look to a concrete example based on a toy model:

$$L=\sqrt{-g}\left[\chi R^n+2\mathcal{L}_m(\daleth,g^{ab})\right]$$

$$\bar{L}=\sqrt{-\bar{g}}\left[\bar{R}-\bar{g}^{ab}\overline{\nabla}_a\phi\overline{\nabla}_b\phi-\boxed{W_0 e^{\sqrt{\frac{2}{3}}\frac{(n-2)}{1-n}\phi}}+2e^{-\frac{\phi}{\sqrt{6}}}\mathcal{L}_m(\daleth,\bar{g}^{ab})\right]$$

The background transforms as

$$S=S_0\left(\frac{t}{t_0}\right)^{\frac{2n}{3(1+w)}} \Rightarrow \begin{cases} \bar{S}=S_0\left(\frac{\bar{t}}{\bar{t}_0}\right)^{\frac{2n}{3(2n-3)(w+1)}} \\ \phi=\phi_0-\frac{1}{(2n-3)\lambda}\ln\left(\frac{\bar{t}^{\,2(n-1)}}{\chi}\right) \end{cases} \quad n\neq\frac{3}{2}$$

$$\begin{cases} \bar{S}=\bar{S}_0(n,\chi)\, e^{\frac{(w+1)\bar{t}}{3\sqrt{6}-6w\chi}} \\ \phi=\phi_0-\frac{(w+1)\bar{t}}{3\sqrt{1-w\chi}} \end{cases} \quad n=\frac{3}{2}$$

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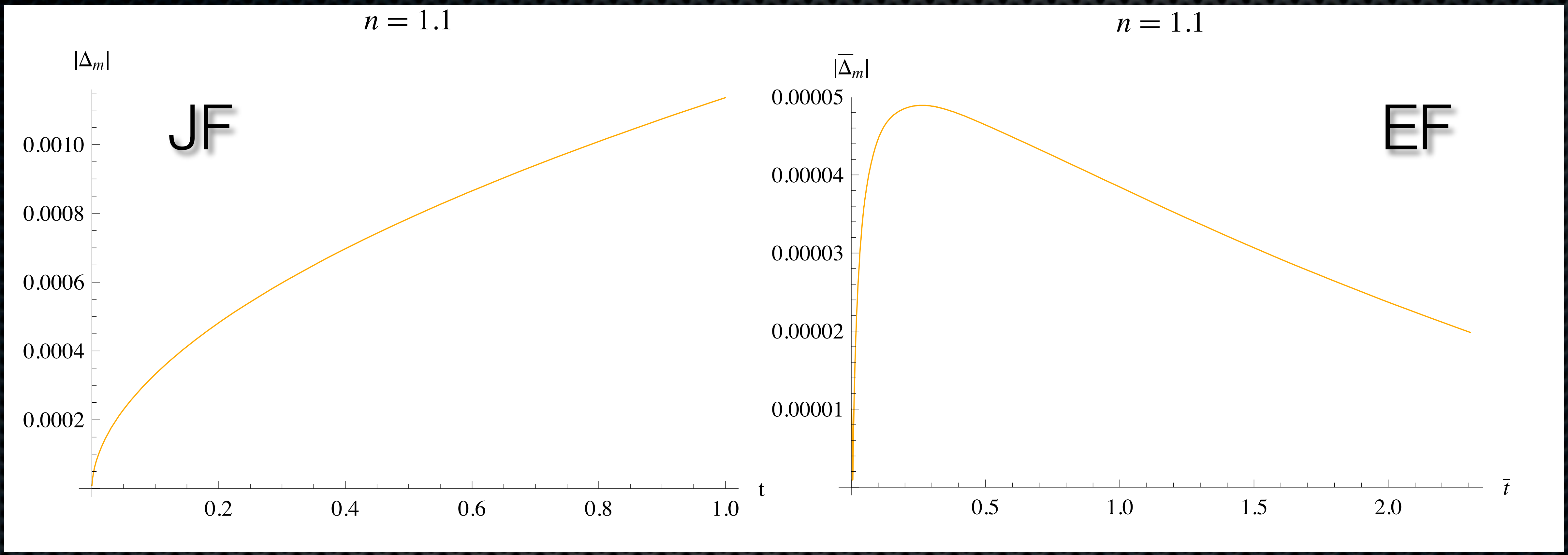
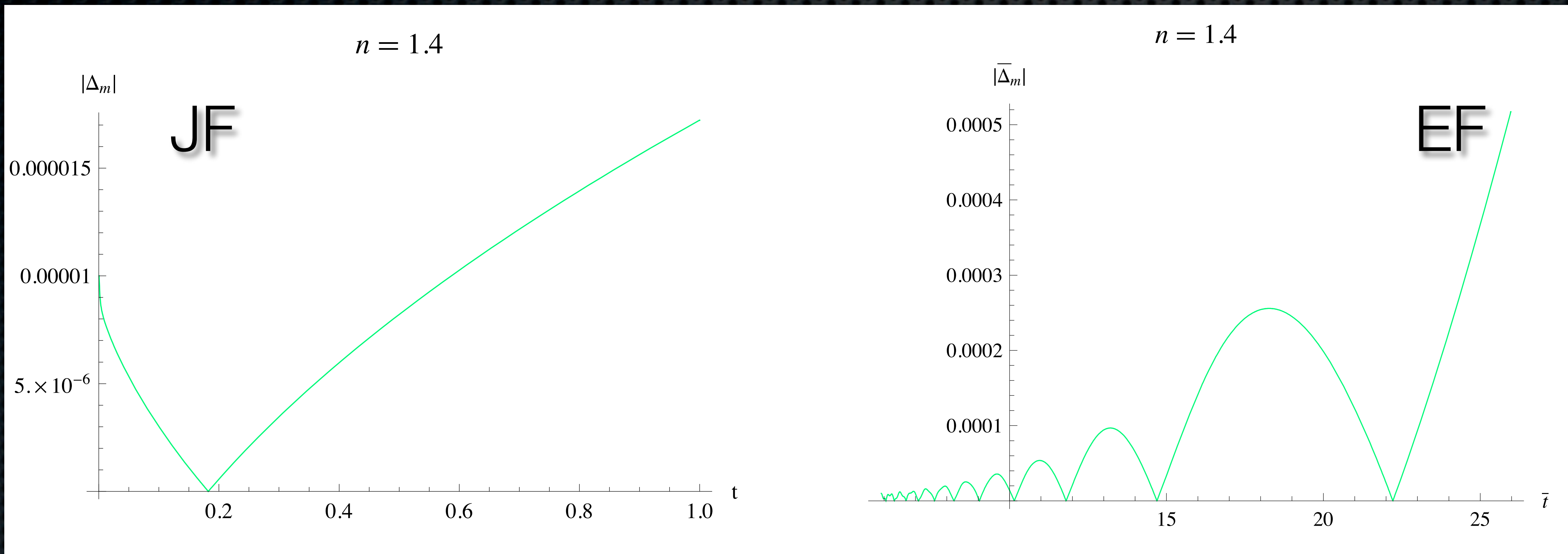
A worked example: R^n-gravity.

and we can compare the behavior of the LW matter fluctuations



there is a difference non only in the rate of growth...

...but also in the presence of oscillations



Conclusions

- We have used the covariant approach to investigate the physics behind the conformal transformations in cosmology
- We have discovered that these transformation can be seen as a change from an inertial to an accelerated observer and a field redefinition.
- The real change in the physics however only take place when one defines the new scalar field.
- We have seen the differences between the Einstein and the Jordan frame at background and perturbations level, discovering the key physical differences.

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