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*The induced Cosmological
Constant as a tool for
exploring geometries in
modified gravity theories*

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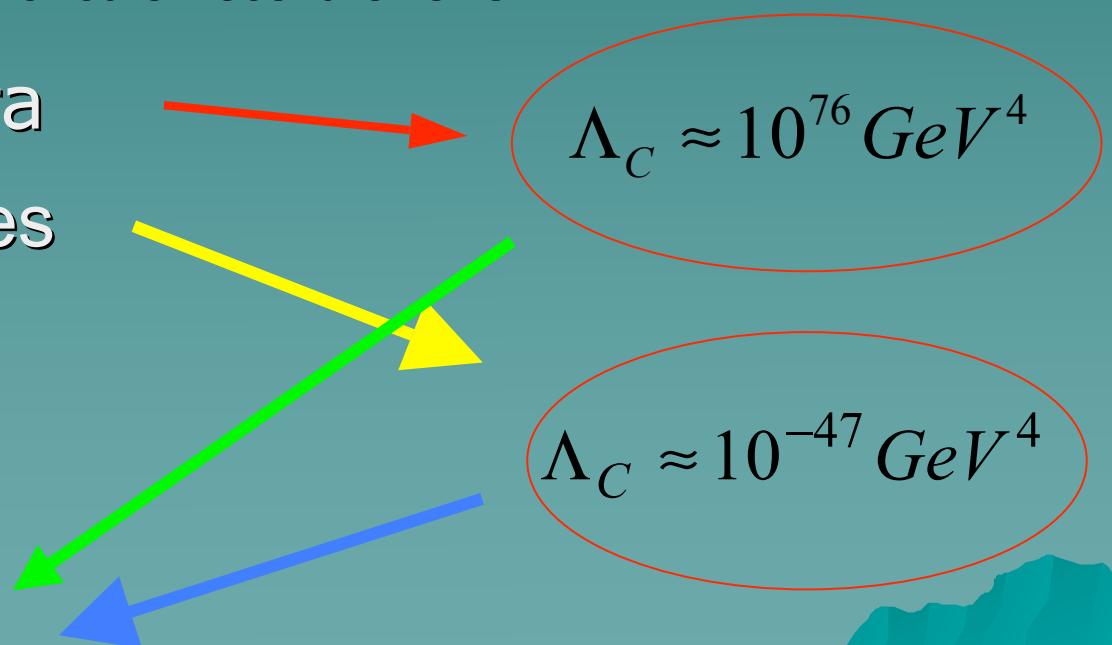
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The Cosmological Constant Problem

For a pioneering review on this problem see S. Weinberg, Rev. Mod. Phys. **61**, 1 (1989). For more recent and detailed reviews see V. Sahni and A. Starobinsky, Int. J. Mod. Phys. D **9**, 373 (2000), astro-ph/9904398; N. Straumann, *The history of the cosmological constant problem* gr-qc/0208027; T.Padmanabhan, Phys.Rept. **380**, 235 (2003), hep-th/0212290. See the ArXiv for other recent reviews...

- ◆ At the Planck era
 - Recent measures



Wheeler-De Witt Equation

B. S. DeWitt, Phys. Rev. 160, 1113 (1967).

$$\left[(2\kappa) G_{ijkl} \pi^{ij} \pi^{kl} - \frac{\sqrt{g}}{2\kappa} (R - 2\Lambda) \right] \Psi[g_{ij}] = 0$$

- ◆ G_{ijkl} is the super-metric, $\kappa=8\pi G$ and Λ is the cosmological constant
- ◆ R is the scalar curvature in 3-dim.
- ◆ Λ can be seen as an eigenvalue
- ◆ $\Psi[g_{ij}]$ can be considered as an eigenfunction

Re-writing the WDW equation

$$\int D[g_{ij}] \Psi^*[g_{ij}] \hat{\Lambda}_\Sigma \Psi[g_{ij}] = \int D[g_{ij}] \Psi^*[g_{ij}] \Lambda(\vec{x}) \Psi[g_{ij}]$$

Where $\hat{\Lambda}_\Sigma = (2\kappa) G_{ijkl} \pi^{ij} \pi^{kl} - \frac{\sqrt{g}}{2\kappa} R$

$$\Lambda(\vec{x}) = -\frac{\sqrt{g}}{\kappa} \Lambda_C$$

Eigenvalue problem

$$\frac{1}{V} \frac{\int D[g_{ij}] \Psi^*[g_{ij}] \int_{\Sigma} d^3x \hat{\Lambda}_{\Sigma} \Psi[g_{ij}]}{\int D[g_{ij}] \Psi^*[g_{ij}] \Psi[g_{ij}]} = -\frac{\Lambda}{\kappa}$$

Solve this infinite dimensional PDE with a Variational Approach

Ψ is a trial wave functional of the gaussian type
Schrödinger Picture

Spectrum of Λ depending on the metric
Energy (Density) Levels

Integration rules on Gaussian wave functionals

$$1 \quad h_{ij}(x) |\Psi\rangle = h_{ij}(x) \Psi[h_{ij}]$$

$$2 \quad \pi_{ij}(x) |\Psi\rangle = -i \frac{\delta}{\delta h_{ij}(x)} \Psi[h_{ij}]$$

$$3 \quad \langle \Psi_1 | \Psi_2 \rangle = \int D[h_{ij}] \Psi_1^*[h_{ij}] \Psi_2[h_{kl}]$$

$$4 \quad \frac{\langle \Psi | h_{ij}(x) |\Psi \rangle}{\langle \Psi | \Psi \rangle} = 0$$

$$5 \quad \frac{\langle \Psi | h_{ij}(x) h_{kl}(y) |\Psi \rangle}{\langle \Psi | \Psi \rangle} = K_{ijkl}(x, y)$$

Form of the background

$$ds^2 = -N^2(r)dt^2 + \frac{dr^2}{1 - \frac{b(r)}{r}} + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

N(r) \leftrightarrow Lapse function
b(r) \leftrightarrow shape function

for example, the Ricci tensor in 3 dim. is

$$R_i^a = \left\{ \frac{b'(r)}{r^2} - \frac{b(r)}{r^3}, \frac{b'(r)}{2r^2} + \frac{b(r)}{2r^3}, \frac{b'(r)}{2r^2} + \frac{b(r)}{2r^3} \right\}$$

Consider $g_{ij} = \bar{g}_{ij} + h_{ij}$

$$\frac{1}{V} \frac{\langle \Psi^* | \int_{\Sigma} d^3x \left[\hat{\Lambda}_{\Sigma}^{(0)} + \hat{\Lambda}_{\Sigma}^{(1)} + \hat{\Lambda}_{\Sigma}^{(2)} + \dots \right] | \Psi \rangle}{\langle \Psi^* | \Psi \rangle} = -\frac{\Lambda}{\kappa}$$

Canonical Decomposition

M. Berger and D. Ebin, J. Diff. Geom. **3**, 379 (1969). J. W. York Jr., J. Math. Phys., **14**, 4 (1973); Ann. Inst. Henri Poincaré **A 21**, 319 (1974).

$$g_{ij} = \bar{g}_{ij} + h_{ij}$$

$$h_{ij} = \frac{1}{3} hg_{ij} + (L\xi)_{ij} + h_{ij}^\perp$$

The canonical decomposition is equivalent to a gauge fixing procedure with the gauge

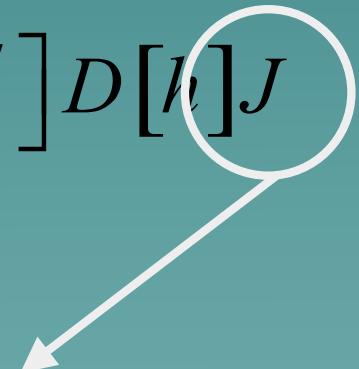
$$\nabla^i \left(h_{ij} - \frac{1}{3} g_{ij} h \right) = 0 = g^{ij} h_{ij}$$

- ◆ h is the trace (spin 0)
- ◆ $(L\xi)_{ij}$ is the gauge part [spin 1 (transverse) + spin 0 (longitudinal)]. It is related to the F.P determinant (ghosts)
- ◆ h_{ij}^\perp represents the transverse-traceless component of the perturbation \rightarrow graviton (spin 2)

Canonical Decomposition

$$\frac{1}{V} \frac{\int D\mu[h] \Psi^*[h] \int_{\Sigma} d^3x \hat{\Lambda}_{\Sigma} \Psi[h]}{\int D\mu[h] \Psi^*[h] \Psi[h]} = -\frac{\Lambda}{\kappa}$$

$$D\mu[h] = D[h_{ij}^\perp] D[\xi_j^T] D[h] J$$



Jacobian related to the F.P. determinant \rightarrow Order by order
ghosts do not contribute in this approach

Graviton Contribution: Regularization

- Zeta function regularization \leftrightarrow Equivalent to the Zero Point Energy subtraction procedure of the Casimir effect

$$\rho_i(\varepsilon, \mu) = \kappa \frac{m_1^4(r)}{16\pi^2} \left[\frac{1}{\varepsilon} \int_{\sqrt{m_i^2(r)}}^{+\infty} \frac{\omega_i^2}{\left(\frac{\mu^2}{\omega_i^2(m)^2(r)} + 1 \right)^{3/2}} d\omega_i - \frac{\ln 2}{2} \right]$$

$$\begin{cases} m_1^2(r) = \frac{6}{r^2} \left(1 - \frac{b(r)}{r} \right) + \frac{3b'(r)}{2r^2} - \frac{3b(r)}{2r^3} \\ m_2^2(r) = \frac{6}{r^2} \left(1 - \frac{b(r)}{r} \right) + \frac{b'(r)}{2r^2} + \frac{3b(r)}{2r^3} \end{cases}$$

Isolating the divergence

$$\begin{aligned}\Lambda &= 8\pi G \left(\underbrace{\rho_1(\varepsilon) + \rho_2(\varepsilon)}_{divergent} + \underbrace{\rho_1(\mu) + \rho_2(\mu)}_{finite} \right) \\ &= \Lambda^{div} + \Lambda^{finite}\end{aligned}$$

$$\Lambda^{div} = \frac{G}{32\pi\varepsilon} (m_1^4(r) + m_2^4(r))$$

Renormalization → Bare cosmological constant changed into

$$\Lambda \rightarrow \Lambda_0 + \Lambda^{div}$$

The finite part becomes

$$\frac{\Lambda_0}{8\pi G} = \rho_1(\mu) + \rho_2(\mu) = \rho_{eff}^{TT}(\mu, r)$$

Eliminate the dependance on μ and impose

$$\frac{1}{8\pi G} \frac{\partial \Lambda_0(\mu)}{\partial \mu} = \mu \frac{d\rho_{eff}^{TT}(\mu, r)}{d\mu}$$

Λ_0 must be treated as running

$$\Lambda_0(\mu, r) = \Lambda_0(\mu_0, r) + \frac{G}{16\pi} (m_1^4(r) + m_2^4(r)) \ln \frac{\mu}{\mu_0}$$

Example: the Schwarzschild Case

- ◆ At the scale μ_0

$$\Lambda_0(\mu_0, r) = -\frac{G}{16\pi} m_0^4(M, r) \ln\left(\frac{m_0^2(M, r)}{4\mu_0^2} \sqrt{e}\right)$$

Λ_0 has a maximum for $\frac{m_0^2(M, r)}{4\mu_0^2} = \frac{1}{e}$ with $\begin{cases} m_1^2(r) \equiv -\frac{3MG}{r^3} = -m_0^2(M, r) \\ m_2^2(r) \equiv +\frac{3MG}{r^3} = +m_0^2(M, r) \end{cases}$ effective mass due to the curvature

$$\frac{\Lambda_0(\mu_0, r)}{8\pi G} = \frac{\mu_0^4}{4\pi e^2} \quad \text{or} \quad \Lambda_0(\mu_0, r) = \frac{1}{64\pi^2} \left(\frac{3MG}{r^3} \right)^2 \quad \text{with } r \in [r_t, 5r_t/4]$$

A Bound on Λ at the scale μ_0

$$\frac{9}{256\pi^2 r_t^4} \leq \frac{\Lambda_0(\mu_0, r)}{8\pi G} \leq \frac{225}{4096\pi^2 r_t^4}$$

A double limit on Λ_0

$$0 \leftarrow \lim_{r \rightarrow r_t} \lim_{r_t \rightarrow 0} \frac{\Lambda_0(\mu_0, r)}{8\pi G} \neq \lim_{r_t \rightarrow 0} \lim_{r \rightarrow r_t} \frac{\Lambda_0(\mu_0, r)}{8\pi G} \rightarrow \infty$$

Signal of Phase Transition???

Example: the dS and AdS Cases

$$b(r) = \pm \frac{\Lambda}{3} r^3 \Rightarrow R = \pm 2\Lambda$$

Adopting the same procedure of the Schwarzschild case, the induced Λ becomes

$$\frac{\Lambda_0(\mu_0, r)}{8\pi G} = -\frac{m_0^4(\Lambda, r)}{64\pi^2} \ln\left(\frac{m_0^2(\Lambda, r)}{4\mu_0^2} \sqrt{e}\right) \text{ with } m_0^2(\Lambda, r) = \frac{6}{r^2} \mp \Lambda$$

Extreme for

$$\frac{\Lambda_0(\mu_0, r)}{8\pi G} = \frac{\mu_0^4}{4\pi e^2} \quad \text{or} \quad \Lambda_0(\mu_0, r) = \frac{1}{64\pi^2} \left(\frac{6}{r^2} \mp \Lambda \right)^2 \quad \text{with} \quad \underbrace{r \in [0, \sqrt{3/\Lambda}]}_{\text{only for dS}}$$

For the dS case on the horizon

$$\frac{m_0^2(\Lambda, r)}{4\mu_0^2} = \frac{1}{e}$$

$$\frac{\Lambda_0(\mu_0, r)}{8\pi G} = -\frac{\Lambda^2}{64\pi^2} \ln\left(\frac{\Lambda\sqrt{e}}{4\mu_0^2}\right) \rightarrow \frac{\Lambda_0(\mu_0, r)}{8\pi G} = \frac{\Lambda^2}{32\pi^2}$$

Extension to f(R) Theories

[S. Capozziello and R.G., Class. Quant. Grav., 24, 1627 (2007)]

- ◆ A straightforward generalization is a f(R) theory substituting the classical Lagrangian with

$$L = \sqrt{-g} (f(R) - 2\Lambda_c) \rightarrow \text{Define } V(P) = \sqrt{g} (f'(R)R - f(R))$$

$$\text{Also define } h(R) = 1 + \frac{2[f'(R) - 1]}{f'(R)}$$

Note → f(R) lives in D+1 dimensions

$$\frac{\Lambda'_0(\mu_0, r)}{8\pi G} = \frac{1}{\sqrt{h(R)}} \left[\frac{\Lambda_0(\mu_0, r)}{8\pi G} + \frac{1}{16\pi GV} \int d^3x \frac{V(P)}{f'(R)} \right]$$

Induced Cosmological Constant modified by f(R)

We can always set $\Lambda'_0(\mu_0, r)$ to zero when

$$\sqrt{h(R)} \frac{\Lambda'_0(\mu_0, r)}{8\pi G} = \frac{1}{16\pi G V} \int d^3x \frac{V(P)}{f'(R)}$$

Explicit Choice: $f(R) = A \exp(-\alpha R)$

$$\frac{\Lambda'_0(\mu_0, r)}{8\pi G} = \sqrt{\frac{A\alpha \exp(-\alpha R)}{3A\alpha \exp(-\alpha R) + 2}} \frac{1}{16\pi G \alpha V} \int d^3x \sqrt{g} (1 + \alpha R)$$

For Schwarzschild $R = 0$

$$\frac{\Lambda'_0(\mu_0, r)}{8\pi G} = \sqrt{\frac{A\alpha}{3A\alpha + 2}} \frac{1}{16\pi G \alpha}$$

$$\frac{1}{64\pi^2} \left(\frac{3MG}{r^3} \right)^2 = \sqrt{\frac{A\alpha}{3A\alpha + 2}} \frac{1}{16\pi G \alpha}$$

$$r^6 > (G\alpha)^2 27r_t^2$$

Explicit Choice: $f(R) = AR^p \exp(-\alpha R)$

$$\frac{\Lambda'_0(\mu_0, r)}{8\pi G} = \sqrt{\frac{A(p - \alpha R)R^{p-1} \exp(-\alpha R)}{A(3p - \alpha R)R^{p-1} \exp(-\alpha R) + 2}} \frac{1}{16\pi G \alpha V} \int d^3x \sqrt{g} \frac{p - \alpha R - 1}{p - \alpha R}$$

For Schwarzschild $R = 0$

$$\frac{\Lambda'_0(\mu_0, r)}{8\pi G} = 0$$

$M = 0 \rightarrow$ No Solution

$f(R) \rightarrow dS$ and AdS cases

$$f(R) = A \exp(-\alpha R) \quad \rightarrow \quad \frac{\Lambda'_0(\mu_0, r)}{8\pi G} = \sqrt{\frac{A\alpha \exp(\mp\alpha 4\Lambda)}{3A\alpha \exp(\mp\alpha 4\Lambda) + 2}} \frac{1 \pm 4\alpha\Lambda}{16\pi G\alpha}$$

for AdS there is a critical value where r.h.s. is zero

$$f(R) = AR^p \exp(-\alpha R)$$

$$\frac{\Lambda'_0(\mu_0, r)}{8\pi G} = \sqrt{\frac{A(p \mp 4\alpha\Lambda)(\pm 4\Lambda)^{p-1} \exp(\mp\alpha 4\Lambda)}{A(3p \mp 4\alpha\Lambda)(\pm 4\Lambda)^{p-1} \exp(\mp\alpha 4\Lambda) + 2}} \frac{1}{16\pi G\alpha} \frac{p \mp 4\alpha\Lambda - 1}{p \mp 4\alpha\Lambda}$$

Currently under investigation!!

Conclusions, Problems and Outlook

- ◆ Wheeler-De Witt Equation \leftrightarrow Sturm-Liouville Problem.
- ◆ The cosmological constant is the eigenvalue.
- ◆ Variational Approach to the eigenvalue equation (infinites).
- ◆ Eigenvalue Regularization with the zeta function \leftrightarrow Casimir energy graviton contribution to the cosmological constant.
- ◆ Renormalization and renormalization group equation. \rightarrow Application to the Maxwell charge.
- ◆ Change the form of $f(R)$
- ◆ Discrete Lichnerowicz spectrum.
- ◆ massive graviton???
- ◆ In progress, spectrum of spherically symmetric metrics