Expanding universes in the conformal frame of $f(R)$ gravity

J. Miritzis, University of the Aegean and R. Giambò, University of Camerino

Invisible Universe 2009, Paris

Outline

- Scalar fields play a prominent role in the construction of cosmological scenarios aiming to describe the evolution of the early and the present universe.

Outline

- Scalar fields play a prominent role in the construction of cosmological scenarios aiming to describe the evolution of the early and the present universe.
- Since the nature of the scalar field supposed to cause accelerated expansion is unknown, it is important to investigate the general properties shared by all FRW models with a scalar field irrespective of the particular choice of the potential.

Outline

- Scalar fields play a prominent role in the construction of cosmological scenarios aiming to describe the evolution of the early and the present universe.
- Since the nature of the scalar field supposed to cause accelerated expansion is unknown, it is important to investigate the general properties shared by all FRW models with a scalar field irrespective of the particular choice of the potential.
- General assumptions:

  1. Initially expanding flat and negatively curved FRW models with a scalar field having an arbitrary bounded from below potential function $V(\phi)$.

Outline

- Scalar fields play a prominent role in the construction of cosmological scenarios aiming to describe the evolution of the early and the present universe.
- Since the nature of the scalar field supposed to cause accelerated expansion is unknown, it is important to investigate the general properties shared by all FRW models with a scalar field irrespective of the particular choice of the potential.
- General assumptions:

1. Initially expanding flat and negatively curved FRW models with a scalar field having an arbitrary bounded from below potential function $V(\phi)$.

2. The scalar field is nonminimally coupled to ordinary matter described by a barotropic fluid with equation of state

$$p = (\gamma - 1)\rho, \quad 0 < \gamma \leq 2.$$

Motivation

- Higher order gravity theories (HOG) derived from Lagrangians of the form

$$L = f(R)\sqrt{-g} + 2L_{\mathrm{m}}(\Psi),$$

f is an arbitrary smooth function and $L_{\mathrm{m}}(\Psi)$ is the matter Lagrangian.

Motivation

- Higher order gravity theories (HOG) derived from Lagrangians of the form

$$L = f(R)\sqrt{-g} + 2L_{\mathrm{m}}(\Psi),$$

f is an arbitrary smooth function and $L_{\mathrm{m}}(\Psi)$ is the matter Lagrangian.

- Under the conformal transformation

$$\widetilde{g}_{\mu\nu} = f'(R)\, g_{\mu\nu},$$

the field equations reduce to the Einstein field equations with a scalar field as an additional matter source

$$\widetilde{G}_{\mu\nu} = T_{\mu\nu}(\widetilde{g}, \phi) + \widetilde{T}_{\mu\nu}(\widetilde{g}, \Psi),$$

where

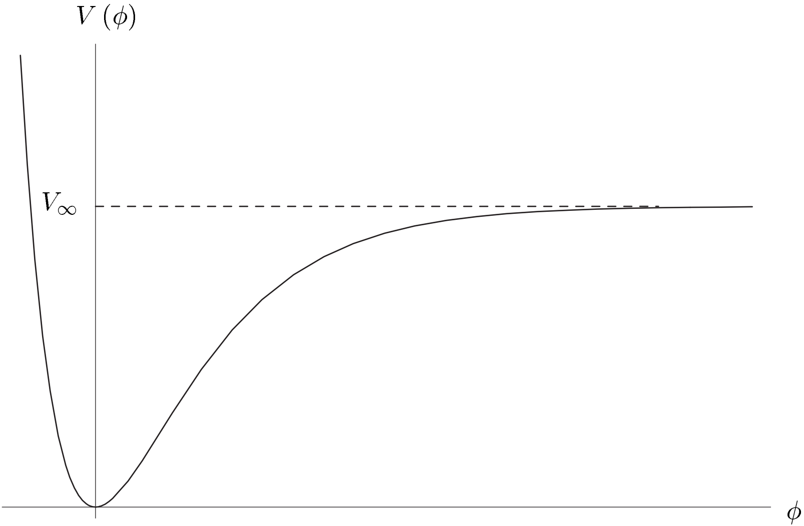
$$T_{\mu\nu}(\widetilde{g}, \phi) = \partial_\mu \phi \partial_\nu \phi - \frac{1}{2}\widetilde{g}_{\mu\nu}\left[(\partial\phi)^2 - 2V(\phi)\right],$$

and

$$\phi = \sqrt{\frac{3}{2}}\ln f'(R), \quad V(R(\phi)) = \frac{1}{2(f')^2}(Rf' - f).$$

Example: Potential arising in the conformal frame of the $R + \alpha R^2$ theory.

$$V(\phi) = \frac{1}{8\alpha}\left(1 - e^{-\sqrt{2/3}\phi}\right)^2$$



Remark

Bianchi identities imply

$$\widetilde{\nabla}^{\mu}\widetilde{T}_{\mu\nu}\left(\widetilde{g},\Psi\right) \neq 0, \quad \widetilde{\nabla}^{\mu}T_{\mu\nu}\left(\widetilde{g},\phi\right) \neq 0,$$

and therefore there is an energy exchange between the scalar field and ordinary matter.

Homogeneous and isotropic spacetimes

- The field equations reduce to the Friedmann equation,

$$H^2 + \frac{k}{a^2} = \frac{1}{3}\left(\rho + \frac{1}{2}\dot{\phi}^2 + V(\phi)\right),$$

Homogeneous and isotropic spacetimes

- The field equations reduce to the Friedmann equation,

$$H^2 + \frac{k}{a^2} = \frac{1}{3}\left(\rho + \frac{1}{2}\dot{\phi}^2 + V(\phi)\right),$$

- the Raychaudhuri equation,

$$\dot{H} = -\frac{1}{2}\dot{\phi}^2 - \frac{\gamma}{2}\rho + \frac{k}{a^2},$$

Homogeneous and isotropic spacetimes

- The field equations reduce to the Friedmann equation,

$$H^2 + \frac{k}{a^2} = \frac{1}{3}\left(\rho + \frac{1}{2}\dot{\phi}^2 + V(\phi)\right),$$

- the Raychaudhuri equation,

$$\dot{H} = -\frac{1}{2}\dot{\phi}^2 - \frac{\gamma}{2}\rho + \frac{k}{a^2},$$

- the equation of motion of the scalar field,

$$\ddot{\phi} + 3H\dot{\phi} + V' = \frac{4-3\gamma}{\sqrt{6}}\rho.$$

Homogeneous and isotropic spacetimes

- The field equations reduce to the Friedmann equation,

$$H^2 + \frac{k}{a^2} = \frac{1}{3}\left(\rho + \frac{1}{2}\dot{\phi}^2 + V(\phi)\right),$$

- the Raychaudhuri equation,

$$\dot{H} = -\frac{1}{2}\dot{\phi}^2 - \frac{\gamma}{2}\rho + \frac{k}{a^2},$$

- the equation of motion of the scalar field,

$$\ddot{\phi} + 3H\dot{\phi} + V' = \frac{4-3\gamma}{\sqrt{6}}\rho.$$

- The Bianchi identities yield the conservation equation,

$$\dot{\rho} + 3\gamma\rho H = -\frac{4-3\gamma}{\sqrt{6}}\rho\dot{\phi},$$

Homogeneous and isotropic spacetimes

- The field equations reduce to the Friedmann equation,

$$H^2 + \frac{k}{a^2} = \frac{1}{3}\left(\rho + \frac{1}{2}\dot{\phi}^2 + V(\phi)\right),$$

- the Raychaudhuri equation,

$$\dot{H} = -\frac{1}{2}\dot{\phi}^2 - \frac{\gamma}{2}\rho + \frac{k}{a^2},$$

- the equation of motion of the scalar field,

$$\ddot{\phi} + 3H\dot{\phi} + V' = \frac{4-3\gamma}{\sqrt{6}}\rho.$$

- The Bianchi identities yield the conservation equation,

$$\dot{\rho} + 3\gamma\rho H = -\frac{4-3\gamma}{\sqrt{6}}\rho\dot{\phi},$$

Set $\frac{4-3\gamma}{\sqrt{6}} = \alpha$

Remarks

- Energy of the scalar field

$$\epsilon = \frac{1}{2}\dot{\phi}^2 + V(\phi) \quad \Rightarrow \quad \dot{\epsilon} = -3H\dot{\phi}^2 + \alpha\rho\dot{\phi}.$$

Remarks

- Energy of the scalar field

$$\epsilon = \frac{1}{2}\dot{\phi}^2 + V(\phi) \quad \Rightarrow \quad \dot{\epsilon} = -3H\dot{\phi}^2 + \alpha\rho\dot{\phi}.$$

- The function

$$W(\phi, \dot{\phi}, \rho, H) = H^2 - \frac{1}{3}\left(\frac{1}{2}\dot{\phi}^2 + V(\phi) + \rho\right),$$

obeys

$$\dot{W} = -2HW.$$

This implies that $\mathrm{sgn}(W)$ is invariant under the flow of the dynamical system.

Remarks

- Energy of the scalar field

$$\epsilon = \frac{1}{2}\dot{\phi}^2 + V(\phi) \quad \Rightarrow \quad \dot{\epsilon} = -3H\dot{\phi}^2 + \alpha\rho\dot{\phi}.$$

- The function

$$W(\phi, \dot{\phi}, \rho, H) = H^2 - \frac{1}{3}\left(\frac{1}{2}\dot{\phi}^2 + V(\phi) + \rho\right),$$

obeys

$$\dot{W} = -2HW.$$

This implies that $\mathrm{sgn}(W)$ is invariant under the flow of the dynamical system.

- The equilibria of the system have the form

$$(\phi = \phi_*, y = 0, \rho = 0, H = \sqrt{V(\phi_*)/3}),$$

where $V'(\phi_*) = 0$.

Asymptotically stable equilibrium

Proposition

Let ϕ_* a strict local minimum for $V(\phi)$, possibly nondegenerate, with nonnegative critical value. Then, $\mathbf{p}_* = (\phi_*, \dot{\phi}_* = 0, \rho_* = 0, H_* = \sqrt{\frac{V(\phi_*)}{3}})$ is an asymptotically stable equilibrium point for expanding cosmologies in the open spatial topologies $k = 0$ and $k = -1$.

Sketch of the proof.

The proof consists in constructing a compact set Ω in $\mathbb{R}^4$ and showing that it is positively invariant. Applying LaSalle's invariance theorem to the functions W and $(\rho + \epsilon)$ in Ω, it is shown that every trajectory in Ω is such that $HW \to 0$ and $H(\dot{\phi}^2 + \gamma\rho) \to 0$ as $t \to +\infty$, which means $\dot{\phi} \to 0,\ \rho \to 0$, and $H^2 - \frac{1}{3}V(\phi) \to 0$. Since H is monotone and admits a limit, $V(\phi)$ also admits a limit, $V(\phi_*)$, thus the solution approaches asymptotically the equilibrium point $\mathbf{p}_*$. $\square$

Asymptotically stable equilibrium

Proposition

Let ϕ_* a strict local minimum for $V(\phi)$, possibly nondegenerate, with nonnegative critical value. Then, $\mathbf{p}_* = (\phi_*, \dot{\phi}_* = 0, \rho_* = 0, H_* = \sqrt{\frac{V(\phi_*)}{3}})$ is an asymptotically stable equilibrium point for expanding cosmologies in the open spatial topologies $k = 0$ and $k = -1$.

Sketch of the proof.

The proof consists in constructing a compact set Ω in $\mathbb{R}^4$ and showing that it is positively invariant. Applying LaSalle's invariance theorem to the functions W and $(\rho + \epsilon)$ in Ω, it is shown that every trajectory in Ω is such that $HW \to 0$ and $H(\dot{\phi}^2 + \gamma\rho) \to 0$ as $t \to +\infty$, which means $\dot{\phi} \to 0,\ \rho \to 0$, and $H^2 - \frac{1}{3}V(\phi) \to 0$. Since H is monotone and admits a limit, $V(\phi)$ also admits a limit, $V(\phi_*)$, thus the solution approaches asymptotically the equilibrium point $\mathbf{p}_*$. □

Similar results were proved in [JM, Class. Quant. Grav. 2003] for separately conserved scalar field and perfect fluid.

Energy exchange

We assume that ϕ_* is a nondegenerate minimum of $V(\phi)$ with null critical value, (for the sake of simplicity we will suppose $\phi_* = 0$)

$$V(\phi) = \frac{1}{2}\omega^2\phi^2 + \mathcal{O}(\phi^3), \qquad \omega > 0.$$

Energy exchange

We assume that ϕ_* is a nondegenerate minimum of $V(\phi)$ with null critical value, (for the sake of simplicity we will suppose $\phi_* = 0$)

$$V(\phi) = \frac{1}{2}\omega^2\phi^2 + \mathcal{O}(\phi^3), \qquad \omega > 0.$$

Suppose that initially the scalar field dominates,

$$\epsilon_0 > \rho_0,$$

and we are asking whether there is a time t_1 such that

$$\rho(t) > \epsilon\,(t)\,, \qquad \forall t > t_1.$$

Energy exchange

We assume that ϕ_* is a nondegenerate minimum of $V(\phi)$ with null critical value, (for the sake of simplicity we will suppose $\phi_* = 0$)

$$V(\phi) = \frac{1}{2}\omega^2\phi^2 + \mathcal{O}(\phi^3), \qquad \omega > 0.$$

Suppose that initially the scalar field dominates,

$$\epsilon_0 > \rho_0,$$

and we are asking whether there is a time t_1 such that

$$\rho(t) > \epsilon(t), \qquad \forall t > t_1.$$

If $V(\phi_*) > 0$, the transition does not happen. In that case, the energy of the scalar field tends to this value, $V(\phi_*) > 0$, whereas the energy of the fluid tends to zero.

Integrating the ρ equation

$$\rho(t) = ce^{-\alpha\phi(t)}a(t)^{-3\gamma} \quad \Rightarrow \quad \rho(t) \simeq ca(t)^{-3\gamma} \quad \text{as} \quad t \to \infty$$

Integrating the ρ equation

$$\rho(t) = ce^{-\alpha\phi(t)}a(t)^{-3\gamma} \;\; \Rightarrow \;\; \rho(t) \simeq ca(t)^{-3\gamma} \quad \text{as} \quad t \longrightarrow \infty$$

The equation of motion of the scalar field,

$$\ddot{\phi} + 3H\dot{\phi} + \omega^2\phi + \mathcal{O}(\phi^2) = \alpha\rho,$$

can be solved by the Kryloff-Bogoliuboff approximation.

Kryloff-Bogoliuboff approximation

$$\ddot{\phi} + \eta f\left(\phi, \dot{\phi}\right) + \omega^2 \phi = 0, \quad 0 < \eta \ll 1.$$

Kryloff-Bogoliuboff approximation

$$\ddot{\phi} + \eta f\left(\phi, \dot{\phi}\right) + \omega^2 \phi = 0, \quad 0 < \eta \ll 1.$$

We are looking for a solution which resembles to the form of the simple harmonic oscillator

$$\begin{aligned} & \phi(t) = A(t) \sin(\omega t + \chi(t)) \\ \text{and} \quad & \dot{\phi}(t) = \omega A(t) \cos(\omega t + \chi(t)). \end{aligned}$$

Kryloff-Bogoliuboff approximation

$$\ddot{\phi} + \eta f\left(\phi, \dot{\phi}\right) + \omega^2 \phi = 0, \quad 0 < \eta \ll 1.$$

We are looking for a solution which resembles to the form of the simple harmonic oscillator

$$\phi(t) = A(t) \sin(\omega t + \chi(t))$$
$$\text{and} \quad \dot{\phi}(t) = \omega A(t) \cos(\omega t + \chi(t)).$$

Setting $\theta(t) = \omega t + \chi(t)$ and substituting into the DE

$$\frac{dA}{dt} = -\frac{\eta}{\omega} f(A \sin\theta, \omega A \cos\theta) \cos\theta,$$
$$\frac{d\chi}{dt} = \frac{\eta}{\omega A} f(A \sin\theta, \omega A \cos\theta) \sin\theta.$$

Replace the RH sides by their *average values* over a range of 2π of θ, i.e. the amplitude $A(t)$ is *regarded as a constant* in taking the average,

$$\frac{dA}{dt} = -\frac{\eta}{2\pi\omega}\int_0^{2\pi} f\left(A\sin\theta, \omega A\cos\theta\right)\cos\theta d\theta,$$

$$\frac{d\chi}{dt} = \frac{\eta}{2\pi\omega A}\int_0^{2\pi} f\left(A\sin\theta, \omega A\cos\theta\right)\sin\theta d\theta.$$

Replace the RH sides by their *average values* over a range of 2π of θ, i.e. the amplitude $A(t)$ is *regarded as a constant* in taking the average,

$$\frac{dA}{dt} = -\frac{\eta}{2\pi\omega}\int_0^{2\pi} f\left(A\sin\theta, \omega A\cos\theta\right)\cos\theta d\theta,$$
$$\frac{d\chi}{dt} = \frac{\eta}{2\pi\omega A}\int_0^{2\pi} f\left(A\sin\theta, \omega A\cos\theta\right)\sin\theta d\theta.$$

Apply the KB approximation to our equation

$$\ddot{\phi} + 3H\dot{\phi} + \omega^2\phi + \mathcal{O}(\phi^2) = \alpha\rho.$$

we find for the amplitude of ϕ

$$\frac{dA}{dt} = -\frac{3}{2}HA + c\frac{\alpha^2 A}{2\omega}a^{-3\gamma} + \mathcal{O}(A^3),$$

Integrating

$$A = Ca^{-3/2} g(t),$$

where $\lim_{t \to \infty} g(t) = 1$, therefore

$$A \simeq a^{-3/2}$$

Integrating

$$A = Ca^{-3/2} g\left(t\right),$$

where $\lim_{t\to\infty} g\left(t\right) = 1$, therefore

$$A \simeq a^{-3/2}$$

Since the amplitudes of ϕ and $\dot{\phi}$ have the same time dependence, $\simeq a^{-3/2}$, we conclude that

$$\epsilon \simeq \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}\omega^2\phi^2 \simeq a^{-3}.$$

Integrating

$$A = Ca^{-3/2} g\left(t\right),$$

where $\lim_{t\to\infty} g\left(t\right) = 1$, therefore

$$A \simeq a^{-3/2}$$

Since the amplitudes of ϕ and $\dot{\phi}$ have the same time dependence, $\simeq a^{-3/2}$, we conclude that

$$\epsilon \simeq \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}\omega^2\phi^2 \simeq a^{-3}.$$

Compare with $\rho \simeq a^{-3\gamma}$.

If $\gamma < 1$ the energy density ρ eventually dominates over the energy density of the scalar field ϵ and this universe follows the classical Friedmannian evolution. For $\gamma > 1$, ϵ eventually dominates over ρ.

Integrating

$$A = Ca^{-3/2} g(t),$$

where $\lim_{t\to\infty} g(t) = 1$, therefore

$$A \simeq a^{-3/2}$$

Since the amplitudes of ϕ and $\dot{\phi}$ have the same time dependence, $\simeq a^{-3/2}$, we conclude that

$$\epsilon \simeq \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}\omega^2\phi^2 \simeq a^{-3}.$$

Compare with $\rho \simeq a^{-3\gamma}$.

If $\gamma < 1$ the energy density ρ eventually dominates over the energy density of the scalar field ϵ and this universe follows the classical Friedmannian evolution. For $\gamma > 1$, ϵ eventually dominates over ρ.

The main obstruction to apply the KB approximation is that the above argument applies only if ρ/A goes to zero. If this does not happen, $\chi(t)$ could in principle be comparable with ωt.