Effect of the Trace Anomaly on the Cosmological Constant

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Invisible Universe

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Outline

- Introduction to the Trace Anomaly
- Dynamically screening the Cosmological Constant: connecting the Cosmological Constant and the Trace Anomaly
- Dynamics Driven by the Trace Anomaly
  - Quasi de Sitter spacetime
  - FLRW spacetimes
- Conclusion

Trace Anomaly

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Einstein Field Equations:

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}\,.$$

Trace Anomaly

Examples:

- Massless, conformally coupled ($\xi = 1/6$) scalar field

$$S = \int \mathrm{d}^4x \sqrt{-g} \left\{ -\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \frac{\xi}{2} R \phi^2 \right\}$$

- Massless spin-1/2 field
- Photon

Trace Anomaly

The trace anomaly or the conformal anomaly in four dimensions is given by:

$$T_{\mathrm{Q}} \equiv \left\langle \hat{T}^{\mu}{}_{\mu} \right\rangle = bF + b' \left(E - \frac{2}{3}\Box R \right) + b''\Box R \, ,$$

where

$$\begin{aligned} E &\equiv {}^{*}R_{\mu\nu\kappa\lambda}{}^{*}R^{\mu\nu\kappa\lambda} = R_{\mu\nu\kappa\lambda}R^{\mu\nu\kappa\lambda} - 4R_{\mu\nu}R^{\mu\nu} + R^2 \\ F &\equiv C_{\mu\nu\kappa\lambda}C^{\mu\nu\kappa\lambda} = R_{\mu\nu\kappa\lambda}R^{\mu\nu\kappa\lambda} - 2R_{\mu\nu}R^{\mu\nu} + \frac{1}{3}R^2 \, , \end{aligned}$$

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where b'' is not fixed uniquely by the trace anomaly and where:

$$\begin{aligned} b &= \frac{1}{120(4\pi)^2}\left(N_S + 6N_F + 12N_V\right) \\ b' &= -\frac{1}{360(4\pi)^2}\left(N_S + \frac{11}{2}N_F + 62N_V\right) . \end{aligned}$$

Trace Anomaly

In (homogenous and isotropic) FLRW spacetimes:

$$g_{\alpha\beta} = \mathrm{diag}\left(-1, a^2(t), a^2(t), a^2(t)\right) , \quad H = \frac{\dot{a}}{a} .$$

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Hence:

$$\begin{aligned} T_{\mathrm{Q}} &= 4b' \left\{ \dddot{H} + 7\ddot{H}H + 4\dot{H}^2 + 18\dot{H}H^2 + 6H^4 \right\} \\ &\quad -6b'' \left\{ \dddot{H} + 7\ddot{H}H + 4\dot{H}^2 + 12\dot{H}H^2 \right\} , \end{aligned}$$

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or, motivated by Ostrogradsky's theorem:

$$T_{\mathrm{Q}} = 24b' \left\{ 3\dot{H}H^2 + H^4 \right\} - 72b''\dot{H}H^2 ,$$

which corresponds to truncating at first order in time derivatives, also known as quasi de Sitter spacetime. Note this is exact when $b'' = 2b'/3$.

Connecting the Trace Anomaly and Cosmological Constant

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- The trace anomaly cannot be generated from a local finite term in the action, but rather stems from a non-local effective action that generates the conformal anomaly by variation with respect to the metric.
- This genuine non-locality of the action generating the trace anomaly reveals a large distance effect of quantum physics. It is then argued that the new conformal field should dynamically screen the cosmological constant, thus solving the cosmological constant problem.

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- Dynamical backreaction should be studied in more general spacetimes than de Sitter spacetime as de Sitter spacetime is not dynamical, i.e.:
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  - In de Sitter spacetime: $H = \mathrm{const}$
  - In FLRW spacetimes: $H = H(t)$

- In the semiclassical spirit, we take expectation values of inhomogeneous quantum fluctuations with respect to a certain state to study its effect on the background spacetime. Therefore, quantum fluctuations affect the background *homogeneously*.

Dynamics Driven by the Trace Anomaly

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- We use covariant stress-energy conservation
- We assume a perfect fluid form for the quantum density and pressure yielding the trace anomaly:

$$T^{\mu}{}_{\nu,\mathrm{Q}} = \left(-\rho_{\mathrm{Q}}, p_{\mathrm{Q}}, p_{\mathrm{Q}}, p_{\mathrm{Q}}\right).$$

Dynamics Driven by the Trace Anomaly

Relevant equation governing the dynamics driven by the trace anomaly is:

$$9(1+\omega)H^2(t)+6\dot{H}(t)-3(1+\omega)\Lambda=-8\pi G\left[T_{\rm Q}+(1-3\omega)\rho_{\rm Q}\right] .$$

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Here, we use either the full or the truncated expression for the quantum trace and the quantum density:

$$\begin{aligned} T_{\rm Q} &= 4b'\left\{\dddot{H} + 7\ddot{H}H + 4\dot{H}^2 + 18\dot{H}H^2 + 6H^4\right\} \\ &\quad -6b''\left\{\dddot{H} + 7\ddot{H}H + 4\dot{H}^2 + 12\dot{H}H^2\right\} , \end{aligned}$$

and:

$$\begin{aligned} \rho_{\rm Q} &= 2b'\left[-2\ddot{H}H + \dot{H}^2 - 6\dot{H}H^2 - 3H^4\right] \\ &\quad +3b''\left[2\ddot{H}H - \dot{H}^2 + 6\dot{H}H^2\right] . \end{aligned}$$

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and:

$$\begin{aligned} \rho_{\mathrm{Q}} &= 2b'\left[\qquad\qquad\qquad - 6\dot{H}H^2 - 3H^4\right] \\ &\quad +3b''\left[\qquad\qquad\qquad + 6\dot{H}H^2\right] . \end{aligned}$$

Dynamics Driven by the Trace Anomaly

We can easily solve for the asymptotic behaviour:

$$\begin{aligned} H_0^{\mathrm{C}} &= \sqrt{\frac{\Lambda}{3}}\left[1 - 8\pi b' \lambda\right] \\ H_0^{\mathrm{A}} &= \sqrt{\frac{-1}{16\pi G\, b'} - \frac{\Lambda}{3}}\,, \end{aligned}$$

where $\lambda = \frac{G\Lambda}{3}$. Note that this asymptotic behaviour does not depend on whether one studies the trace anomaly in quasi de Sitter or in FLRW spacetimes.

Dynamics in Quasi de Sitter Spacetime

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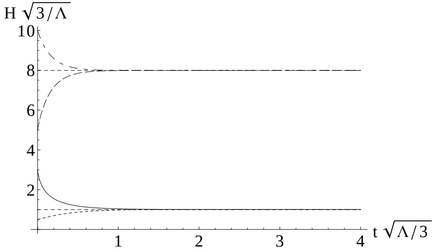
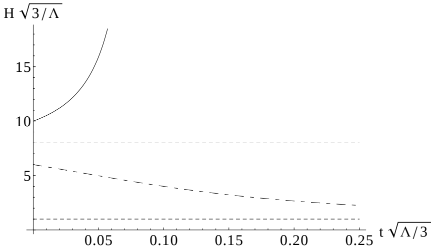
- Moreover, by linearising around the quantum anomaly driven attractor:

$$\delta(t) = \frac{H(t) - H_0^{\mathrm{A}}}{H_0^{\mathrm{A}}} \,,$$

one finds a region in parameter space where this attractor becomes unstable.

Dynamics in Quasi de Sitter Spacetime





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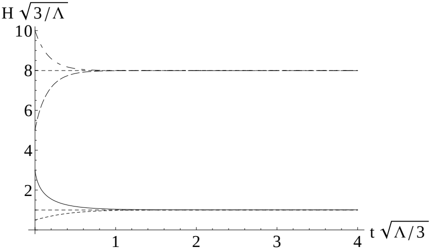

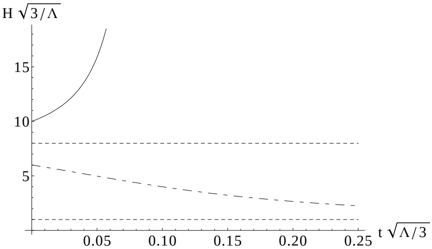

Adding the trace anomaly does not change the effective value of the cosmological constant in quasi de Sitter spacetime.

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- One can again linearise the Einstein Field equations around H_0^{A} and H_0^{C} and derive a stability condition on the attractors:

$$\text{If } b'' - 2b'/3 > 0, \qquad \text{then} \begin{cases} \text{Classical attractor unstable} \\ \text{Quantum attractor stable} \end{cases}$$

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- Note that when $b'' - 2b'/3 = 0$, we return to the quasi de Sitter spacetime analysis performed before.

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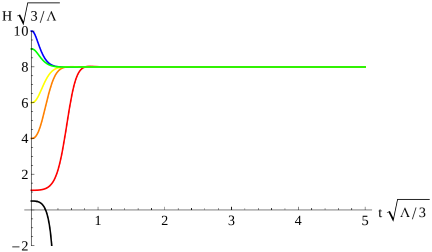

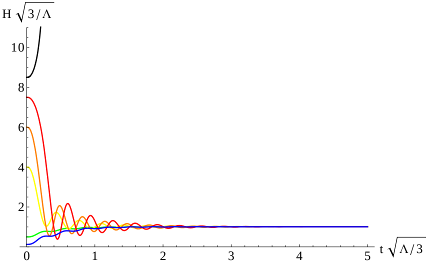

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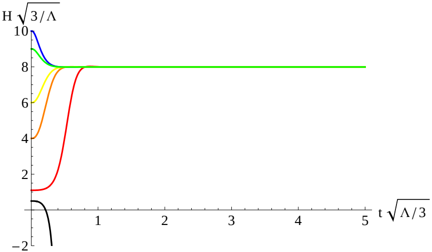

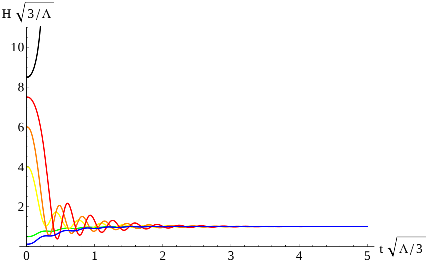

If the classical de Sitter attractor is stable, adding the trace anomaly does not change its effective value in FLRW spacetimes.

Dynamics in FLRW Spacetimes

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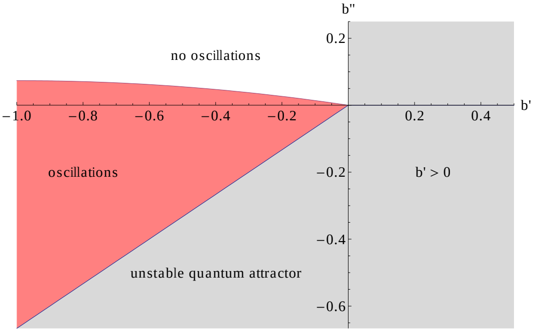
Oscillatory behaviour occurs whenever the eigenvalues of the small perturbations around the two attractors develop an imaginary contribution. We find:

- Whenever the classical attractor is stable, oscillations occur
- When the quantum anomaly driven attractor is stable, oscillations occur when:

$$b'' < -\frac{2}{9} b' \left(\frac{1 + 8\pi\lambda b'}{1 + 8\pi\lambda} \right) .$$

Dynamics in FLRW Spacetimes



Conclusion

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- We have studied the dynamics of the Hubble parameter both in quasi de Sitter and in FLRW spacetimes including matter, a cosmological term and the trace anomaly.
- There is no dynamical effect that influences the effective value of the cosmological constant, i.e.: the classical de Sitter attractor.
- Based on our semiclassical analysis we thus conclude that the trace anomaly does *not* solve the cosmological constant problem.

The Einstein-Hilbert action reads:

$$S = S_{\rm EH} + S_{\rm M} = \frac{1}{16\pi G}\int d^4x\sqrt{-g}\,(R - 2\Lambda) + \int d^4x\sqrt{-g}\mathcal{L}_{\rm M}\,,$$

where:

$$\mathcal{L}_{\rm M} = -\frac{1}{2}\partial_\alpha\phi(x)\partial_\beta\phi(x)g^{\alpha\beta} - \frac{1}{2}m^2\phi^2(x) - V(\phi(x))\,.$$

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The Einstein Field Equations follow as usual:

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}\,.$$

Dynamics Driven by the Trace Anomaly

Quantum contribution to the stress-energy tensor is:

$$
\begin{aligned}
T_{\mu\nu} &= -\frac{2}{\sqrt{-g}}\frac{\delta}{\delta g^{\mu\nu}}\Gamma[\phi_{\rm cl}] = -\frac{2}{\sqrt{-g}}\frac{\delta}{\delta g^{\mu\nu}}\left(S_{\rm M}[\phi_{\rm cl}] + \Gamma_{\rm Q}[\phi_{\rm cl}]\right) \\
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Classically, we have $\nabla^{\mu} T^{\rm C}_{\mu\nu} = 0$, hence we derive:

$$\nabla^{\mu} T^{\rm Q}_{\mu\nu} = 0\,.$$

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When writing:

$$T^{\mu}{}_{\nu,\mathrm{Q}} = (-\rho_{\mathrm{Q}}, p_{\mathrm{Q}}, p_{\mathrm{Q}}, p_{\mathrm{Q}})\,,$$

we derive from stress-energy conservation for the quantum contributions:

$$\begin{aligned}\rho_{\mathrm{Q}} \quad &= \quad 2b' \left[-2\ddot{H}H + \dot{H}^2 - 6\dot{H}H^2 - 3H^4\right] \\ &\quad +3b'' \left[2\ddot{H}H - \dot{H}^2 + 6\dot{H}H^2\right]\,.\end{aligned}$$

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Or, truncated at order $\epsilon = -\frac{\dot{H}}{H^2}$:

$$\rho_{\mathrm{Q}} = -6b'\left[2\dot{H}H^2 + H^4\right] + 18b''\dot{H}H^2 .$$

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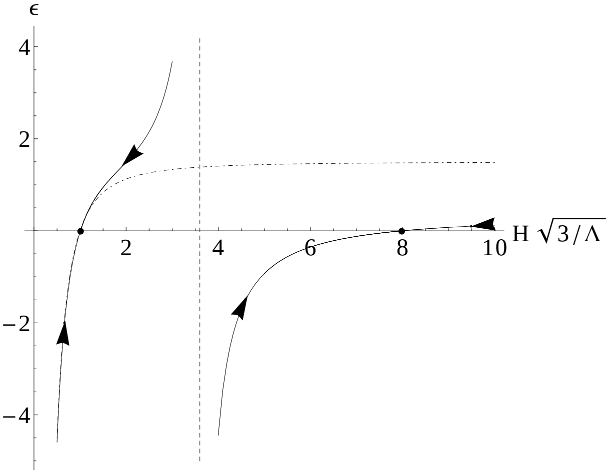
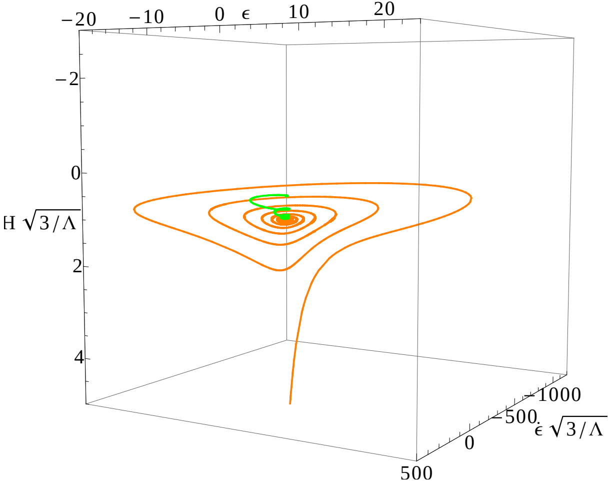
We use the (00) Einstein equation to eliminate ρ_{M}:

$$R_{00} - \frac{1}{2} R g_{00} + \Lambda g_{00} = 8\pi G \left(\rho_{\mathrm{M}} + \rho_{\mathrm{Q}} \right) \, .$$

Phase space structure in quasi de Sitter spacetime



Phase space structure in FLRW spacetime



Phase space structure in FLRW spacetime